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that a relation between these surfaces, which initially is *entirely absent*, is supplied by multiplying the number of sides of the prism, is to deceive ourselves.

Equally vulnerable seems the corollary commonly attached, that the lateral area of the cylinder is therefore the limit of the lateral area of the inscribed prism when the number of sides is indefinitely increased; for such a corollary presupposes that we have already defined the *area* of a *curved surface* in some other way.

Such unjustifiable assumptions may be avoided by first showing that the area of the inscribed or circumscribed prism has a limit when the number of sides is indefinitely increased, and then *defining* the area of the cylinder as the limit of the area of the inscribed or circumscribed cylinder.

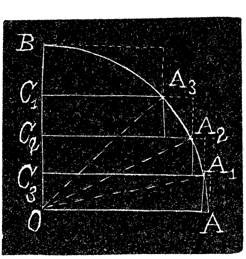
THE VOLUME OF THE SPHERE.

By HENRY L. COAR, A. M., University of Illinois.

The solids of revolution, that is, the solids generated by the revolution of a plane figure about an axis, offer many interesting problems to the student of synthetic geometry of three dimensions. The question of obtaining the volumes and total areas of solids, generated by the revolution of rectilinear figures, presents no particular difficulty, but when we attempt to find by purely synthetic means the volumes and superficial area of solids formed by curvilinear figures, the problem is not always simple without making some assumptions regarding limits. A well-known problem of this kind is to find the volume and area of an anchor-ring. A most interesting problem in this line is that of obtaining the volume of a sphere regarded as a solid of revolution. In the following proof, which I have not been able to find published anywhere, no assumptions of any kind regarding the existence of the limits are necessary.

We need consider only the hemisphere, which is generated by the revolution, through 360°, of a quadrant of a circle about a radius.

Let AOB be the quadrant of a circle and let us divide the radius OB into n equal parts. Then construct a set of inscribed and a set of circumscribed rectangles as indicated in the figure.



If now we rotate the complete figure through 360° about the radius OB, the quadrant will generate a hemisphere, while each of the inscribed as well as each of the circumscribed rectangles will generate a right circular cylinder. Let us designate the sum of the volumes of the cylinders generated by the inscribed rectangles by V_1, that of the cylinders generated by the circumscribed rectangles by V_2, and the volume of the hemisphere by V. We will first prove that both V_1 and V_2 approach V as their limit as n increases indefinitely, and will then find an expression for either of these and obtain its limiting value. To prove the first we have always $V_1 < V < V_2$, hence

$$V_2 - V < V_2 - V_1 \text{ and } V - V_1 < V_2 - V_1.$$

But $V_2 - V_1$ is the volume of the lowest circumscribed cylinder, *i. e.*

$$V_2 - V_1 = \frac{r}{n}.\pi r^2 = \frac{\pi r^3}{n}$$

where r is the radius of the circle. Hence

$$V_2 - V < \frac{\pi r^3}{n} \text{ and } V - V_1 < \frac{\pi r^3}{n}.$$

It follows therefore that each of these differences can be made less than any assignable positive quantity, and hence

$$\lim_{n \doteq \infty} V_1 = V \text{ and } \lim_{n \doteq \infty} V_2 = V.$$

We thus prove that our hemisphere is actually the limit of the figures in question.

Let us now obtain an expression, say for V_2, and find its limiting value We see that V_2 is the sum of n right circular cylinders. Let us find their volumes and add. For convenience denote the volumes of these n cylinders, begining with the lowest one, by $v_1, v_2, \ldots . v_n$. We shall then have

$$v_1 = \frac{r}{n}.AO^2.\pi = \frac{\pi r^3}{n},$$

$$v_2 = \frac{r}{n}.A_1C_1^2.\pi = \frac{\pi r}{n}\left(r^2 - \frac{r^2}{n^2}\right) = \frac{\pi r^3}{n}\left(1 - \frac{1}{n^2}\right),$$

$$v_3 = \frac{r}{n}.A_2C_2^2.\pi = \frac{\pi r}{n}\left[r^2 - \left(\frac{2r}{n}\right)^2\right] = \frac{\pi r^3}{n}\left(1 - \frac{2^2}{n^2}\right),$$

. . . .

. . . .

$$v_n = \frac{r}{n}(A_{n-1}C_{n-1})^2.\pi = \frac{\pi r}{n}\left[r^2 - \left(\frac{\overline{n-1}\,r}{n}\right)^2\right] = \frac{\pi r^3}{n}\left(1 - \frac{(n-1)^2}{n^2}\right).$$

Hence

$$V_2 = v_1 + v_2 + \ldots . + v_n = \frac{\pi r^3}{n}\left[1 + \left(1 - \frac{1}{n^2}\right) + \left(1 - \frac{2^2}{n^2}\right) + \left(1 - \frac{3^2}{n^2}\right) + \ldots .\right.$$

$$+\left(1-\frac{(n-1)^2}{n^2}\right)\Bigg]$$

$$=\frac{\pi r^3}{n}\left[n-\left(\frac{1}{n}\right)^2-\left(\frac{2}{n}\right)^2-\left(\frac{3}{n}\right)^2-\dots\dots-\left(\frac{n-1}{n}\right)^2\right]$$

$$=\pi r^3\left\{1-\frac{1}{n^3}[1^2+2^2+3^2+\dots\dots+(n-1)^8]\right\}.$$

Now the series $1^2+2^2+3^2+\dots\dots+(n-1)^2$ is the sum of the squares of the first $n-1$ positive integers. The formula for the sum of the squares of the first n integers is given in any College Algebra under "piles of shot" and is

$$1^2+2^2+3^2+\dots\dots+n^2=\frac{n(n+1)\,(2n+1)}{6}.$$

Hence

$$1^2+2^2+3^2+\dots\dots+(n-1)^2=\frac{(n-1)n\,(2n-1)}{6}.$$

Substituting this in the expression for V_2 we have

$$V_2=\pi r^3\left[1-\frac{1}{n^3}.\frac{(n-1)n\,(2n-1)}{6}\right]$$

$$V_2=\pi r^3\left[1-\tfrac{1}{6}(1-\frac{1}{n}).1.(2-\frac{1}{n})\right].$$

Now proceed to the limit and we have

$$V=\lim_{n\doteq\infty}V_2=\pi r^3[1-\tfrac{1}{6}(1).1.(2)]=\pi r^3[1-\tfrac{1}{3}]=\tfrac{2}{3}\pi r^3$$

which is the well-known result.

DERIVATION OF FORMULA FOR Tan ½A IN SPHERICAL TRIGONOMETRY.

By GEORGE R. DEAN, School of Mines, Rolla, Mo.

Applying Napier's Rules of Circular Parts to the triangle formed by the bisector of angle A, the radius of the inscribed circle and one of the sides, we see that $\sin(s-a)=\tan r\cot\frac{1}{2}A$. Hence, $\tan\frac{1}{2}A=\tan r/\sin(s-a)$, so that